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The linear meson model and chiral perturbation theory*

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Abstract. We compare the linear meson model and chiral perturbation theory in next to leading order in the quark mass expansion. In particular, we compute the couplings L_4–L_8 of chiral perturbation theory as functions of the parameters of the linear model. They are induced by the exchange of 0^{++} scalar mesons. We use a phenomenological analysis of the effective vertices of the linear model in terms of pseudoscalar meson masses and decay constants. Our results for the L_i agree with previous phenomenological estimates.

1 Introduction

Spontaneously broken chiral symmetry is a basic ingredient for the description of light mesons in the context of QCD. It is based on the observation that in the limit of vanishing light quark masses, $m_u, m_d, m_s \to 0$, the QCD Lagrangian exhibits a chiral $SU_L(3) \times SU_R(3)$ flavor invariance. Since in this limit the light meson spectrum only shows an explicit vector–like $SU_V(3)$ symmetry the full chiral flavor invariance must be broken spontaneously by a "chiral condensate". For small quark masses m_q ($q = u, d, s$ here) one expects the existence of some sort of systematic expansion[1] in powers of m_q. To a given order in m_q the spontaneously broken chiral symmetry only allows for a finite number of effective couplings entering the description of the pseudoscalar octet (π, K, η). To lowest order these are the pion decay constant F_0 and the proportionality constant B_0 between the squared meson masses and the quark masses, e.g., $M^2_{\pi^\pm} = B_0(m_u + m_d)$. With increasing order in m_q the number of independent chiral invariants and therefore of free parameters grows rapidly. Nevertheless, useful information can be extracted from the low orders of the quark mass expansion if m_q is sufficiently small.

There are different ways of realizing the symmetry content of QCD within low energy models for the light mesons. The minimal model is the nonlinear sigma model for the pseudoscalar octet which is described by a special unitary 3×3 matrix $\tilde{U}$, $\tilde{U}^\dagger \tilde{U} = 1$, $\det \tilde{U} = 1$ [1,2]. It can be extended to include the η' meson explicitly if the constraint $\det \tilde{U} = 1$ is dropped. In the linear sigma model [3] the low lying scalar and pseudoscalar mesons are grouped into a field Φ transforming in the $(\overline{\mathbf{3}}, \mathbf{3})$ representation of the chiral group [4,5]. Here the additional 0^{++} octet describes the scalar mesons (a_0, K_0^*, f_0) and the 0^{++} singlet corresponds to the "σ–resonance". Further extensions of mesonic models also include fields for the lightest vector and pseudovector mesons (see, e.g., [6,5] and references therein). A projection of these models onto the pseudoscalar octet states can be computed by first integrating the mesonic quantum fluctuations and subsequently accounting for the exchange of the η', the scalar, vector and pseudovector mesons (depending on the model) by solving their field equations. Technically, all masses and vertices are contained in the *effective action*[2] for the mesons which is the generating functional for the $1PI$ Green functions. All quantum fluctuations are already included in the effective action. (Of course, the concept of one–particle–irreducibility depends on the particle content of the model. For example, a four pion interaction mediated by the exchange of a ρ–meson becomes $1PI$ in the pure nonlinear or linear sigma model.) As far as only the pseudoscalar octet (or nonet) is concerned a common language for the different mesonic models can be found by computing the effective action restricted to the corresponding fields. Formally, the latter can be obtained from the effective action of the linear model by solving the field equations for the scalar fields as functions of the pseudoscalar fields and inserting the result into the effective action. (A similar procedure can be applied to models which contain fields for the vector and axial–vector mesons.) We emphasize that the symmetry content of all these mesonic models is the same as far as the pseudoscalar octet is concerned. All relations that follow from spontaneously broken chiral symmetry if the masses and interactions of the pseudoscalars are computed to a given order in m_q are identical. This

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[1] Not all quantities are analytic in m_q, though. For instance, the pion mass is proportional to $(m_u + m_d)^{1/2}$. Further non–analyticities in higher couplings are induced by quantum fluctuations of massless Goldstone bosons for $m_q \to 0$

[2] Often the terms "effective model" or "effective action" are also used to describe the result of a reduction of degrees of freedom. In this case we will rather employ here the term "low energy model" and reserve the term "effective action" for the generating functional of $1PI$ Green functions

also holds if the quark mass expansion is connected to a simultaneous expansion in powers of derivatives for the pseudoscalar fields as usually done in chiral perturbation theory. The quark mass expansion is, however, not unique. Instead of truncating the computation of physical quantities at a given order in m_q one may alternatively choose to include in the effective action only those invariants which contribute to a given order in m_q. If no further truncations are performed the physical observables calculated from these invariants contain contributions which are formally of higher order in m_q. This procedure has been followed in [5] for the linear meson model. It amounts to a partial resummation of the quark mass expansion for physical quantities. The results obtained from the linear and the nonlinear model may therefore differ by higher order quark mass effects. Beyond pure symmetry relations the different models typically contain additional assumptions which may lead to further predictive power for mesonic observables.

Chiral perturbation theory describes the "Goldstone degrees of freedom" by a nonlinear sigma model. It is valid for momenta sufficiently below the ρ–meson and σ–resonance masses. There the parameters F_0, B_0 as well as the couplings L_i which appear at next to leading order in the quark mass expansion are specified at some normalization scale μ. The propagators and vertices of the effective action or the physical observables are then computed using a loop expansion which accounts for the fluctuations of the pseudoscalar mesons. This expansion apparently converges if the momenta and μ are not too large [2]. The most important implicit assumption is simply that higher orders in the quark mass expansion can be neglected. Since this is the minimal model pure symmetry relations are most easily visualized in this context.

Within the linear meson model one may assume that a description in terms of the field Φ and quark degrees of freedom holds up to a "compositeness scale" k_Φ around $(600-700)\,\mathrm{MeV}$. (In addition, one may use explicit vector fields.) If, furthermore, one assumes that the Yukawa coupling between Φ and the quarks is sufficiently strong at the scale k_Φ this model turns out to be very predictive for the effective mesonic action. This is a consequence of a fast evolution towards partial infrared fixed points of its running couplings [7]. We will follow here a more modest "phenomenological" approach [5] and concentrate only on properties of the effective action for the linear meson model. The main assumption made here are the ones of reference [5]:

(i) Terms with more than two derivatives can be neglected in the effective action if the latter is expanded in powers of differences of momenta from an appropriately chosen average momentum of the corresponding $SU_V(3)$ multiplet, like $q_0^2 = -(2M_{K^\pm}^2 + M_{\pi^\pm}^2)/3$ for the pseudoscalar octet. This assumption should hold with the exception of contributions from the (tree–)exchange of vector or axial–vector fields. It is substantiated by the apparent smallness of the momentum dependence of effective couplings which is induced by meson loops [5]. Furthermore, we neglect (as in [5]) certain higher order invariants with two derivatives involving high powers of the chiral condensate σ_0.

(ii) The explicit flavor symmetry breaking by current quark masses appears in the linear meson model only in form of a *linear* source term

$$\mathcal{L}_\jmath = -\frac{1}{2}\,\mathrm{Tr}\left(\Phi^\dagger \jmath + \jmath^\dagger \Phi\right) \tag{1.1}$$

$$\jmath = \jmath^\dagger = a_q M_q \equiv a_q \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} . \tag{1.2}$$

This is motivated by the way how mesons arise as quark–antiquark composite states at the scale k_Φ [8, 9]. Deviations from this assumption arise from current quark mass dependencies of quark loop contributions to the effective four–quark vertex at the compositeness scale. Since typical momenta are larger or of the order of k_Φ these effects are small.

(iii) Invariants in the effective action which contribute only beyond quadratic order in the quark mass expansion can be neglected in the phenomenological analysis. We emphasize here that the quark mass expansion apparently converges faster in the linear than in the nonlinear model [5]. This is related to effective resummations which are particularly important in the case of large mixing of states.

It is important to note that our assumptions do not restrict the effective action of the linear meson model to renormalizable interactions. The appearance of "non–renormalizable interactions", i.e. couplings with negative mass dimension, is, of course, not only possible but even necessary for the generating functional of $1PI$ Green functions of the linear meson model. It is, however, not clear a priori how large the corresponding couplings are in units of the chiral condensate σ_0. It was found in [5,10] that such non–renormalizable interactions play a crucial role for the compatibility of this model with phenomenology. They can be explained to a large extent by mixing with or exchange of higher mass states.

The purpose of this work is a comparison of the effective action of chiral perturbation theory within the framework of the nonlinear sigma model on one side with that of the linear meson model on the other side. Naturally, such a comparison must be restricted to the pseudoscalar fields as the only degrees of freedom present in the nonlinear model. Both models have free parameters which can be fixed by using experimental input. As far as the same quantities are used as input — the meson masses $M_{\pi^\pm}$, $M_{K^\pm}$ and M_η as well as the decay constants f_π and f_K — and only symmetry relations are employed there is, of course, no difference between the models. In the linear model we also use $M_{\eta'}$ and the decay constants f_η and $f_{\eta'}$ as obtained from the decays $\eta \to 2\gamma$, $\eta' \to 2\gamma$ (related to singlet–octet mixing) for a determination[3] of the

[3] Actually, only two of the three observables f_η, $f_{\eta'}$ and M_η are needed as input whereas the third one comes out as a rather successful "prediction" of the linear model [5]

parameters [5] relevant for the present work. From this information we determine the couplings L_i which appear in next to leading order of chiral perturbation theory. More precisely, we will compute within the linear model effective couplings $\tilde{L}_i$ which multiply proper vertices with the same structure as those appearing as interactions in next to leading order in chiral perturbation theory. In the linear model the effective derivative terms are evaluated for momenta corresponding to the location of the pole for an average pseudoscalar octet mass[4] $q_0^2 = -(2M_{K^\pm}^2 + M_{\pi^\pm}^2)/3$. For a full comparison one should evaluate in chiral perturbation theory the same effective vertices. For simplicity we will identify here the $\tilde{L}_i$ with the couplings $L_i(\mu)$ of chiral perturbation theory, evaluated at a renormalization scale $\mu^2 = -q_0^2$. This minimizes the logarithms in the loop expansion of chiral perturbation theory. Nevertheless, for a quantitatively precise comparison the same $1PI$ vertices would have to be computed in the framework of chiral perturbation theory.

Recent estimates of the current quark masses in both the nonlinear [11] and the linear [10] sigma model show very good agreement. The estimate of the higher order couplings L_i presented here confirms this general picture. Within errors, the results of the linear model are compatible with the estimates from chiral perturbation theory. For $\mu = 400\,\mathrm{MeV}$ the couplings $\tilde{L}_5$ and $\tilde{L}_8$ of the linear meson model turn out to be somewhat smaller than the central values of the couplings $L_5(\mu)$ and $L_8(\mu)$, respectively, in chiral perturbation theory. This may partly be due to the uncertainties in the optimal value for μ or the loop effects which are neglected in our comparison. On the other hand, it was observed earlier [2,5] that the quark mass expansion converges relatively slowly for the nonlinear model. It seems therefore also conceivable that the observed differences reflect higher order quark mass corrections which are neglected in next to leading order chiral perturbation theory but are included in our analysis of the linear model.

In the linear sigma model the scalar and pseudoscalar meson fields are represented by a complex 3×3 matrix Φ. The effective potential can be constructed from the invariants

$$\begin{aligned} \rho &= \mathrm{Tr}\, \Phi^\dagger \Phi \quad , \quad \xi = \det \Phi + \det \Phi^\dagger \\ \tau_2 &= \frac{3}{2} \mathrm{Tr} \left(\Phi^\dagger \Phi - \frac{1}{3}\rho \right)^2 , \; \tau_3 = \mathrm{Tr} \left(\Phi^\dagger \Phi - \frac{1}{3}\rho \right)^2 . \end{aligned} \tag{1.3}$$

We use a polynomial expansion around their values for the expectation value of Φ in presence of equal light current quark masses,

$$\langle \Phi \rangle = \overline{\sigma}_0 \mathbf{1} . \tag{1.4}$$

With $\rho_0 = 3\overline{\sigma}_0^2$ and $\xi_0 = 2\overline{\sigma}_0^3$ it is parameterized by

$$\begin{aligned} V &= \overline{m}_g^2 (\rho - \rho_0) - \frac{1}{2}\overline{\nu} \left[\xi - \xi_0 - \overline{\sigma}_0 (\rho - \rho_0) \right] \\ &+ \frac{1}{2}\overline{\lambda}_1 (\rho - \rho_0)^2 + \frac{1}{2}\overline{\lambda}_2 \tau_2 + \frac{1}{2}\overline{\lambda}_3 \tau_3 \\ &+ \frac{1}{2}\overline{\beta}_1 (\rho - \rho_0)(\xi - \xi_0) \\ &+ \frac{1}{2}\overline{\beta}_2 (\rho - \rho_0)\tau_2 + \frac{1}{2}\overline{\beta}_3 (\xi - \xi_0)\tau_2 + \frac{1}{2}\overline{\beta}_4 (\xi - \xi_0)^2 \\ &+ \ldots . \end{aligned} \tag{1.5}$$

[4] Throughout this article we work in Euclidean space–time. This is most convenient for comparison of our results with those of an exact renormalization group approach [7]–[9]

The mass term $\overline{m}_g^2$ vanishes for zero quark masses

$$\overline{m}_g^2 = \frac{1}{6\overline{\sigma}_0} \mathrm{Tr}\, \jmath . \tag{1.6}$$

Beyond the minimal kinetic term $\sim Z_\Phi \,\mathrm{Tr}\, \partial_\mu \Phi^\dagger \partial^\mu \Phi$ the effective action contains more complicated terms involving two derivatives of Φ multiplied by couplings X_Φ^-, U_Φ, $\tilde{V}_\Phi$, etc. [5]. They play an important role for the phenomenological analysis and will be specified in detail in the next section. As a consequence, the wave function renormalization constants for the different $SU_V(3)$ multiplets contained in Φ are different, as for instance Z_m and Z_p for the pseudoscalar octet and singlet, respectively,

$$\begin{aligned} Z_m &= Z_\Phi + X_\Phi^- \overline{\sigma}_0^2 + U_\Phi \overline{\sigma}_0 \\ Z_p &= Z_\Phi + X_\Phi^- \overline{\sigma}_0^2 + 6\tilde{V}_\Phi \overline{\sigma}_0^4 - 2U_\Phi \overline{\sigma}_0 . \end{aligned} \tag{1.7}$$

Because of the axial anomaly reflected by the couplings $\overline{\nu}$, $\overline{\beta}_1$, $\overline{\beta}_3$ and $\overline{\beta}_4$ the pseudoscalar singlet acquires a mass term even for vanishing current quark masses

$$m_p^2 = \left(\frac{3}{2}\overline{\nu}\,\overline{\sigma}_0 + \overline{m}_g^2 \right) Z_p^{-1} . \tag{1.8}$$

For a phenomenological determination of the couplings $\overline{\nu}$, $\overline{\lambda}_2$, etc. (or the related renormalized parameters $\sigma_0 = Z_m^{-1/2}\overline{\sigma}_0$, $\nu = Z_m^{-3/2}(\overline{\nu} + \ldots)$, $\lambda_2 = Z_m^{-2}\overline{\lambda}_2 + \ldots$) we refer the reader to [5]. We only recall here that $\overline{\nu}$ is essentially determined from the η'–mass, $\overline{\sigma}_0$ and $\overline{\lambda}_2$ are related to f_π and $f_K - f_\pi$ whereas the non–minimal kinetic couplings are fixed by the decays $\eta \to 2\gamma$ and $\eta' \to 2\gamma$. Information from the scalar sector is not needed for a determination of these parameters. It will turn out that the couplings $\tilde{L}_5$ and $\tilde{L}_8$ are generated by the exchange of 0^{++} octet fields and one therefore expects that the average scalar octet mass $m_h^2 = (2M_{K_o^*}^2 + M_{a_o}^2)/3$ plays a decisive role. It will turn out, however, that the combination $\frac{Z_h}{Z_m} m_h^2$ which appears in these couplings (with Z_h the wave function renormalization constant of the scalar octet) only involves parameters which can be determined from the pseudoscalar sector alone (at least to the required order in the quark mass expansion):

$$\frac{Z_h}{Z_m} m_h^2 \simeq \frac{2}{3} \frac{Z_p}{Z_m} M_{\eta'}^2 + 3\lambda_2 \sigma_0^2 . \tag{1.9}$$

The couplings $\tilde{L}_4$ and $\tilde{L}_6$ need, in addition, information on the octet–singlet mass splitting in the scalar sector which has not yet been determined phenomenologically. The coupling $\tilde{L}_7 = -\sigma_0/(18\nu)$ is essentially fixed by f_π and the η' mass. Not much new information is obtained for the higher derivative couplings $\tilde{L}_1$, $\tilde{L}_2$ and $\tilde{L}_3$. They are dominated by corresponding higher derivative terms in the linear model which are, in turn, mainly generated by the exchange of vector mesons.

We finally want to mention an earlier analysis [12] of the hypothesis that the couplings L_i are dominated by the exchange of scalar and vector mesons. Our findings are in overall agreement with this work. Beyond [12] we determine here the relevant couplings and wave function renormalization for the scalar octet. This permits a rather precise quantitative estimate of the low energy couplings $\tilde{L}_4$, $\tilde{L}_5$, $\tilde{L}_6$, $\tilde{L}_7$ and $\tilde{L}_8$ within our framework. Possible effects of higher states and loops are already included in the effective action for the linear meson model. Within the above assumptions (i), (ii), (iii) there are no further corrections to these couplings. The low energy constants L_i have also been estimated within extended Nambu–Jona-Lasinio models [13].

2 Nonlinear versus linear sigma model

Our starting point is the polar decomposition of the general complex 3×3 matrix Φ in terms of a hermitian matrix S and a unitary matrix U

$$\Phi = SU \; , \quad S^\dagger = S \; , \quad U^\dagger U = \mathbf{1} \; . \tag{2.1}$$

Here U contains the nine 0^{-+} pseudoscalars in a nonlinear representation

$$U = \exp\left\{-\frac{i}{3}\vartheta\right\}\tilde{U}, \;\; \det\tilde{U} = 1, \;\; \tilde{U} = \exp\left\{\frac{i\lambda_z \Pi_z}{f}\right\} \tag{2.2}$$

with λ_z the Gell–Mann matrices in a normalization $\mathrm{tr}\,\lambda_z\lambda_y = 2\delta_{zy}$ and $f \equiv (2f_K + f_\pi)/3$ the average pseudoscalar decay constant. In case of chiral symmetry breaking the matrix U describes the nine Goldstone bosons which would be massless in the absence of quark masses and the chiral anomaly. It therefore transforms under $U_L(3)\times U_R(3)$ chiral flavor transformations according to

$$U \longrightarrow \mathcal{U}_R U \mathcal{U}_L^\dagger \; . \tag{2.3}$$

With respect to these transformations Φ belongs to a linear $(\overline{\mathbf{3}}, \mathbf{3})$ representation and this determines the transformation properties of $S = \Phi U^\dagger$ as

$$\Phi \longrightarrow \mathcal{U}_R \Phi \mathcal{U}_L^\dagger \; , \quad S \longrightarrow \mathcal{U}_R S \mathcal{U}_R^\dagger \; . \tag{2.4}$$

Thus S is neutral with respect to $SU_L(3)$ and the Abelian axial $U_A(1)$ symmetry. It decomposes with respect to $SU_R(3)$ into an octet (the traceless part) and a singlet ($\sim \mathrm{tr}\, S$). The nine real degrees of freedom contained in S describe in a nonlinear representation the nine 0^{++} scalar fields contained in Φ. With respect to the discrete symmetries $\mathcal{C}$ and $\mathcal{P}$ the fields transform according to

$$\begin{aligned} \mathcal{P} &: \Phi \to \Phi^\dagger \;, \; U \to U^\dagger \;, \; S \to U^\dagger S U \\ \mathcal{C} &: \Phi \to \Phi^T \;, \; U \to U^T \;, \; S \to U^T S^T U^* \; . \end{aligned} \tag{2.5}$$

It is easy to visualize the group theoretical properties underlying the ansatz (2.1) by observing that an arbitrary complex matrix Φ can be brought into a diagonal and real form $\hat{\Phi} = \mathrm{diag}(\hat{\Phi}_u, \hat{\Phi}_d, \hat{\Phi}_s)$ by suitable $U_L(3)\times U_R(3)$ transformations $\Phi = \hat{\mathcal{U}}_R \hat{\Phi} \hat{\mathcal{U}}_L^\dagger$. With $U = \hat{\mathcal{U}}_R \hat{\mathcal{U}}_L^\dagger$ we can write $\Phi = \hat{\mathcal{U}}_R \hat{\Phi} \hat{\mathcal{U}}_R^\dagger U$ and associate $S = \hat{\mathcal{U}}_R \hat{\Phi} \hat{\mathcal{U}}_R^\dagger = S^\dagger$. We finally observe that the decomposition (2.1) is not unique. By a redefinition $S \to SV^{-1}$, $U \to VU$ the properties (2.1) remain unchanged if V is unitary and obeys $VSV = S$. The possible solutions for V depend on S. For instance, $S = 1$ requires $V^2 = 1$ whereas $S = 0$ is trivially compatible with all V. For generic S there is only a discrete number of solutions V. We conclude that for given S we should define U only modulo V. We can use this freedom to require that for $\Phi = \Phi^\dagger$ one has $U = 1$. In particular, in the presence of a real diagonal source for Φ (corresponding to a real diagonal quark mass matrix) the expectation value of Φ is also real and diagonal such that $\langle\Phi\rangle = \langle S\rangle$, $\langle U\rangle = 1$.

We can now express the different pieces in the chirally invariant effective Lagrangian for Φ in terms of U and S. The four independent invariants (1.3) on which the potential V, (1.5), depends are given by

$$\begin{aligned} \rho &= \mathrm{Tr}\, S^2, \qquad \xi = 2\det S\cos\vartheta, \\ \tau_2 &= \frac{3}{2}\,\mathrm{Tr}\left(S^2 - \frac{1}{3}\,\mathrm{Tr}\, S^2\right)^2, \\ \tau_3 &= \mathrm{Tr}\left(S^2 - \frac{1}{3}\,\mathrm{Tr}\, S^2\right)^3 . \end{aligned} \tag{2.6}$$

The additional invariant $\omega = i(\det\Phi - \det\Phi^\dagger)$ is $\mathcal{CP}$–odd and may therefore only appear quadratically. Yet, ω^2 can be expressed in terms of the invariants (2.6). The potential depends therefore on S and the pseudoscalar singlet field described by ϑ.

For the simplest form of the effective kinetic term one finds

$$\begin{aligned} \mathcal{L}_{\mathrm{kin}}(Z_\Phi) &= Z_\Phi\,\mathrm{Tr}\left(\partial^\mu\Phi^\dagger\partial_\mu\Phi\right) \\ &= Z_\Phi\Big\{\mathrm{Tr}\left(\partial^\mu S\partial_\mu S\right) + \mathrm{Tr}\left(S^2\partial^\mu U\partial_\mu U^\dagger\right) \\ &\quad + \mathrm{Tr}\left([S,\partial^\mu S]U\partial_\mu U^\dagger\right)\Big\} \end{aligned} \tag{2.7}$$

whereas some non–minimal kinetic invariants introduced in [5] read

$$\begin{aligned} \mathcal{L}_{\mathrm{kin}}(Y_\Phi) &= \frac{1}{4}Y_\Phi\partial_\mu\rho\partial^\mu\rho \\ &= Y_\Phi\,\mathrm{Tr}\left(S\partial_\mu S\right)\mathrm{Tr}\left(S\partial^\mu S\right) \\ \mathcal{L}_{\mathrm{kin}}(V_\Phi) &= \frac{1}{2}V_\Phi\partial^\mu\xi\partial_\mu\xi \\ &= 2V_\Phi\Big\{\cos^2\vartheta\,\partial^\mu\det S\,\partial_\mu\det S \\ &\quad - 2\det S\cos\vartheta\sin\vartheta\,\partial^\mu\det S\,\partial_\mu\vartheta \end{aligned}$$

$$+ (\det S)^2 \sin^2\vartheta\partial^\mu\vartheta\partial_\mu\vartheta \Big\}$$

$$\begin{aligned}\mathcal{L}_{\rm kin}(\tilde{V}_\Phi) = \frac{1}{2}\tilde{V}_\Phi\partial^\mu\omega\partial_\mu\omega = 2\tilde{V}_\Phi\Big\{ &\sin^2\vartheta\partial^\mu \det S\partial_\mu \det S \\ &+ 2\det S\cos\vartheta\sin\vartheta\partial^\mu \det S\partial_\mu\vartheta \\ &+ (\det S)^2\cos^2\vartheta\partial^\mu\vartheta\partial_\mu\vartheta\Big\} \end{aligned} \tag{2.8}$$

$$\begin{aligned}&\mathcal{L}_{\rm kin}(X_\Phi^\pm) \\ &= -\frac{1}{8}X_\Phi^\pm\Big\{\mathrm{Tr}\left(\Phi^\dagger\partial_\mu\Phi \pm \partial_\mu\Phi^\dagger\Phi\right)\left(\Phi^\dagger\partial^\mu\Phi \pm \partial^\mu\Phi^\dagger\Phi\right) \\ &\quad + \mathrm{Tr}\left(\Phi\partial_\mu\Phi^\dagger \pm \partial_\mu\Phi\Phi^\dagger\right)\left(\Phi\partial^\mu\Phi^\dagger \pm \partial^\mu\Phi\Phi^\dagger\right)\Big\} \\ &= -\frac{1}{4}X_\Phi^\pm\Big\{2\,\mathrm{Tr}\left(S\partial^\mu SS\partial_\mu S \pm S^2\partial^\mu S\partial_\mu S\right) \\ &\quad + \mathrm{Tr}\left\{\left((2\mp 1)\left[S\partial^\mu SS, S\right] \mp \left[S^3, \partial^\mu S\right]\right)\partial_\mu UU^\dagger\right\} \\ &\quad - \mathrm{Tr}\left\{(2\mp 1)\,S^2U\partial^\mu U^\dagger S^2\partial_\mu UU^\dagger \right. \\ &\quad \left. \mp S^4\partial^\mu U\partial_\mu U^\dagger\right\}\Big\}. \end{aligned}$$

We also include an "anomalous" $U_A(1)$ violating kinetic term involving ϵ–tensors

$$\begin{aligned}\mathcal{L}_{\rm kin}(U_\Phi) &= \frac{1}{2}U_\Phi\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\left(\Phi_{a_1b_1}\partial^\mu\Phi_{a_2b_2}\partial_\mu\Phi_{a_3b_3}\right. \\ &\qquad \left. +\Phi^\dagger_{a_1b_1}\partial^\mu\Phi^\dagger_{a_2b_2}\partial_\mu\Phi^\dagger_{a_3b_3}\right) \\ &= U_\Phi\cos\vartheta\Big\{\mathrm{Tr}\,S\,\mathrm{Tr}\,\partial_\mu S\,\mathrm{Tr}\,\partial^\mu S + 2\,\mathrm{Tr}\left(S\partial_\mu S\partial^\mu S\right) \\ &\quad - \mathrm{Tr}\,S\,\mathrm{Tr}\left(\partial_\mu S\partial^\mu S\right) - 2\,\mathrm{Tr}\,\partial_\mu S\,\mathrm{Tr}\left(S\partial^\mu S\right) \\ &\quad + \mathrm{Tr}\left(\partial_\mu S\left[S^2, \partial^\mu UU^\dagger\right]\right) \\ &\quad - \mathrm{Tr}\,S\,\mathrm{Tr}\left(\partial_\mu S\left[S, \partial^\mu UU^\dagger\right]\right) \\ &\quad + \det S\left(\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right) - \partial_\mu\vartheta\partial^\mu\vartheta\right)\Big\} \\ &\quad - iU_\Phi\sin\vartheta\Big\{2\,\mathrm{Tr}\,S\,\mathrm{Tr}\,\partial_\mu S\,\mathrm{Tr}\left(S\partial^\mu UU^\dagger\right) \\ &\quad - \mathrm{Tr}\,S\,\mathrm{Tr}\left(\partial_\mu S\left\{S, \partial^\mu UU^\dagger\right\}\right) \\ &\quad - 2\,\mathrm{Tr}\,\partial_\mu S\,\mathrm{Tr}\left(S^2\partial^\mu UU^\dagger\right) \\ &\quad - 2\,\mathrm{Tr}\left(\partial_\mu UU^\dagger S\right)\mathrm{Tr}\left(S\partial^\mu S\right) \\ &\quad + \mathrm{Tr}\left(\partial_\mu S\left(S\left\{S, \partial^\mu UU^\dagger\right\} + \left\{S, \partial^\mu UU^\dagger\right\}S\right)\right)\Big\} \end{aligned} \tag{2.9}$$

where we used the relation

$$\begin{aligned}&\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}A_{a_1b_1}B_{a_2b_2}C_{a_3b_3} = \\ &\mathrm{Tr}\,A\,\mathrm{Tr}\,B\,\mathrm{Tr}\,C + \mathrm{Tr}\left(A\left\{B, C\right\}\right) \\ &- \mathrm{Tr}\,A\,\mathrm{Tr}\,(BC) - \mathrm{Tr}\,B\,\mathrm{Tr}\,(AC) - \mathrm{Tr}\,C\,\mathrm{Tr}\,(AB)\ . \end{aligned} \tag{2.10}$$

In addition, there are terms involving four and more derivatives of Φ.

The effective action of the nonlinear sigma model obtains if we restrict the degrees of freedom of the linear model to the pseudoscalars in the nonlinear realization U. A straightforward approach inserts (2.1), (2.2) into the potential V and the kinetic terms of the linear meson model. Subsequently one expresses S as a functional of U by means of the solution of its equation of motion. This is, of course a rather difficult task. On the other hand, we are only interested in the effective action for U or $\tilde{U}$ to low orders in an expansion in powers of derivatives and M_q. This reduces the complexity of this program considerably.

3 Lowest order chiral perturbation theory

We will start by considering first the lowest order of the quark mass expansion of the effective action for the linear meson model. This coincides with lowest order chiral perturbation theory. In this approximation Φ can be written as

$$\Phi = \varphi_{00}U\ , \quad \text{i.e.} \quad S = \varphi_{00}\mathbf{1}\ . \tag{3.1}$$

Here the positive real constant φ_{00} should be associated with the expectation value of Φ in the limit of vanishing quark masses. Only in this limit φ_{00} coincides with $\overline{\sigma}_0$ and $\langle\Phi\rangle - \varphi_{00} = 0$, whereas for $m_q \neq 0$ there are corrections involving the trace of the quark mass matrix.

The computation of the parameters of chiral perturbation theory from those of the linear meson model proceeds in two steps: First one inserts (3.1) with (2.2) into the kinetic terms (2.7)–(2.9), the potential (1.5) and the source term (1.1) linear in the quark masses. Second, one establishes the relation between φ_{00} and $\overline{\sigma}_0$. This yields an effective action for $\tilde{U}$ and ϑ in the limit of small quark masses. Within our approach the parameters in this effective action correspond to an expansion in momenta around a typical momentum in the vicinity of the pole for the pseudoscalar mesons, i.e. $q_0^2 = -(2M^2_{K^\pm} + M^2_{\pi^\pm})/3$.

Let us start with the kinetic terms which yield

$$\begin{aligned}\mathcal{L}^{(0)}_{\rm kin} &= \left(Z_\Phi + U_\Phi\varphi_{00}\cos\vartheta + X_\Phi^-\varphi_{00}^2\right)\varphi_{00}^2\,\mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U \\ &\quad + \varphi_{00}^3\partial^\mu\vartheta\partial_\mu\vartheta\left(2\varphi_{00}^3V_\Phi\sin^2\vartheta + 2\varphi_{00}^3\tilde{V}_\Phi\cos^2\vartheta\right. \\ &\qquad \left. -U_\Phi\cos\vartheta\right)\ . \end{aligned} \tag{3.2}$$

With the identities

$$\begin{aligned}&\mathrm{Tr}\left(\tilde{U}^\dagger\partial_\mu\tilde{U}\right) = 0\ , \quad \mathrm{Tr}\left(\partial_\mu UU^\dagger\right) = -i\partial_\mu\vartheta \\ &\mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U = \mathrm{Tr}\,\partial^\mu\tilde{U}^\dagger\partial_\mu\tilde{U} + \frac{1}{3}\partial^\mu\vartheta\partial_\mu\vartheta \end{aligned} \tag{3.3}$$

we find the singlet kinetic term from (3.2) for

$$\begin{aligned}\mathcal{L}^{(0)}_{\rm kin}(\vartheta) = \frac{1}{3}\varphi_{00}^2\partial^\mu\vartheta\partial_\mu\vartheta\Big\{&Z_\Phi + X_\Phi^-\varphi_{00}^2 - 2U_\Phi\varphi_{00}\cos\vartheta \\ &+ 6\varphi_{00}^4\left(V_\Phi\sin^2\vartheta + \tilde{V}_\Phi\cos^2\vartheta\right)\Big\}\ . \end{aligned} \tag{3.4}$$

We note that at $\vartheta = 0$ the kinetic term is positive provided $Z_p^{(0)} = Z_\Phi + X_\Phi^- \varphi_{00}^2 - 2U_\Phi\varphi_{00} + 6\tilde{V}_\Phi\varphi_{00}^4 > 0$. We recognize the wave function renormalization Z_p, (1.7), up to the difference between φ_{00} and $\overline{\sigma}_0$. Similarly, the combination $Z_m^{(0)} = Z_\Phi + U_\Phi\varphi_{00} + X_\Phi^-\varphi_{00}^2$ in front of the octet kinetic term $\sim \operatorname{Tr}\partial^\mu\tilde{U}^\dagger\partial_\mu\tilde{U}$ in (3.2) is the lowest order approximation of Z_m. (1.7). Instead of ϑ we may use a field p with standard normalization of the kinetic term

$$\frac{1}{2}\partial^\mu p\partial_\mu p = \frac{1}{3}\varphi_{00}^2 Z_p \partial^\mu\vartheta\partial_\mu\vartheta \,. \tag{3.5}$$

With (3.1) the effective potential V (1.5) is independent of $\tilde{U}$ and contributes only a potential for ϑ

$$\begin{aligned}V(\vartheta) &= -\frac{1}{2}\left[\overline{\nu} + 3\overline{\beta}_1(\overline{\sigma}_0^2 - \varphi_{00}^2) + 2\overline{\beta}_4\overline{\sigma}_0^3\right]\xi(\vartheta) + \frac{1}{2}\overline{\beta}_4\xi(\vartheta)^2\\ \xi(\vartheta) &= 2\varphi_{00}^3\cos\vartheta \,.\end{aligned} \tag{3.6}$$

Expanding around the minimum at $\vartheta = 0$ this induces a mass term for the pseudoscalar singlet

$$\begin{aligned}\frac{1}{2}\overline{M}_\vartheta^2\vartheta^2 &= \frac{1}{2}\varphi_{00}^3\left[\overline{\nu} + 3\overline{\beta}_1(\overline{\sigma}_0^2 - \varphi_{00}^2) + 2\overline{\beta}_4(\overline{\sigma}_0^3 - 2\varphi_{00}^3)\right]\vartheta^2\\ &= \frac{1}{2}p^2 Z_p^{-1}\Big[\frac{3}{2}\overline{\nu}\varphi_{00} + \frac{9}{2}\overline{\beta}_1\varphi_{00}(\overline{\sigma}_0^2 - \varphi_{00}^2)\\ &\qquad + 3\overline{\beta}_4\varphi_{00}(\overline{\sigma}_0^3 - 2\varphi_{00}^3)\Big]\\ &= \frac{1}{2}m_p^{(0)2}p^2 \,.\end{aligned} \tag{3.7}$$

Here $m_p^{(0)}$ can be identified to lowest order with m_p, (1.8) or the mass of the η' meson. For vanishing quark masses we therefore have a massless octet of Goldstone bosons $\tilde{U}$ and a massive pseudoscalar singlet p. The source term (1.1)

$$\begin{aligned}\mathcal{L}_\jmath^{(0)} &= -\frac{1}{2}\varphi_{00}\operatorname{Tr}\left(\jmath^\dagger U + U^\dagger\jmath\right)\\ &= -\frac{1}{2}a_q\varphi_{00}\operatorname{Tr}M_q\left(U + U^\dagger\right)\end{aligned} \tag{3.8}$$

generates non–vanishing masses for the pseudoscalar octet.

It is now straightforward to identify the parameters of the nonlinear sigma model to lowest order in the quark mass expansion. We use here the notation of [2] adapted to Euclidean space–time

$$\begin{aligned}\mathcal{L}_{\chi\mathrm{PT}}^{(0)} &= \frac{\tilde{F}_0^2}{4}\left\{\operatorname{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right) - 2\tilde{B}_0\operatorname{Tr}M_q\left(U + U^\dagger\right)\right\}\\ &\quad + \frac{1}{12}\tilde{H}_0\partial^\mu\vartheta\partial_\mu\vartheta \,.\end{aligned} \tag{3.9}$$

We find

$$\begin{aligned}\tilde{F}_0^2 &= 4\varphi_{00}^2\left(Z_\Phi + U_\Phi\varphi_{00} + X_\Phi^-\varphi_{00}^2\right)\\ \tilde{B}_0 &= \frac{1}{4}\frac{a_q}{\left(Z_\Phi + U_\Phi\varphi_{00} + X_\Phi^-\varphi_{00}^2\right)\varphi_{00}}\\ \tilde{H}_0 &= -12U_\Phi\varphi_{00}^3 + 24\tilde{V}_\Phi\varphi_{00}^6\end{aligned} \tag{3.10}$$

and f in (2.2) is given to this order by

$$f = 2Z_m^{1/2}\varphi_{00} \,. \tag{3.11}$$

The constants $\tilde{F}_0^2$ and $\tilde{H}_0$ can be computed from the parameters of the linear meson model once the relation between φ_{00} and $\overline{\sigma}_0$ is established. To lowest order in the quark mass expansion we can use $\varphi_{00} = \overline{\sigma}_0 = Z_m^{-1/2}(2f_K + f_\pi)/6$. For quantitative estimates in next to leading order we also need to include the difference between φ_{00} and $\overline{\sigma}_0$. The value of φ_{00} corresponds to the minimum of the effective potential (1.5) without the source term. We only need to evaluate V for diagonal fields $\Phi = \overline{\sigma}_0 + \Delta\sigma$

$$\begin{aligned}V(\Delta\sigma) &= 6\overline{m}_g^2\overline{\sigma}_0\Delta\sigma\\ &+ \left(3\overline{m}_g^2 - \frac{3}{2}\overline{\nu}\,\overline{\sigma}_0 + 18\overline{\lambda}_1\overline{\sigma}_0^2 + 18\overline{\beta}_1\overline{\sigma}_0^3 + 18\overline{\beta}_4\overline{\sigma}_0^4\right)\Delta\sigma^2\\ &+ \left(-\overline{\nu} + 18\overline{\lambda}_1\overline{\sigma}_0 + 27\overline{\beta}_1\overline{\sigma}_0^2 + 36\overline{\beta}_4\overline{\sigma}_0^3\right)\Delta\sigma^3\\ &+ \left(\frac{9}{2}\overline{\lambda}_1 + 15\overline{\beta}_1\overline{\sigma}_0 + 30\overline{\beta}_4\overline{\sigma}_0^2\right)\Delta\sigma^4\\ &+ \left(3\overline{\beta}_1 + 12\overline{\beta}_4\overline{\sigma}_0\right)\Delta\sigma^5 + 2\overline{\beta}_4\Delta\sigma^6 \,.\end{aligned} \tag{3.12}$$

Including corrections to quadratic order in $\Delta\sigma$ this yields

$$\begin{aligned}\varphi_{00} &= \overline{\sigma}_0 - \overline{\sigma}_0\frac{\overline{m}_g^2}{\overline{m}_s^2}\\ &\quad + \frac{1}{2}\frac{\overline{m}_g^4\overline{\sigma}_0^2}{\overline{m}_s^6}\left(\overline{\nu} - 18\overline{\lambda}_1\overline{\sigma}_0 - 27\overline{\beta}_1\overline{\sigma}_0^2 - 36\overline{\beta}_4\overline{\sigma}_0^3\right)\end{aligned} \tag{3.13}$$

where

$$\overline{m}_s^2 = \overline{m}_g^2 - \frac{1}{2}\overline{\nu}\,\overline{\sigma}_0 + 6\overline{\lambda}_1\overline{\sigma}_0^2 + 6\overline{\beta}_1\overline{\sigma}_0^3 + 6\overline{\beta}_4\overline{\sigma}_0^4 \tag{3.14}$$

equals the scalar singlet mass term m_s^2 up to a wave function renormalization constant ($m_s^2 = \overline{m}_s^2/Z_s$). Inserting (3.13) into (3.10) one can now extract the difference between $\tilde{F}_0$ and $f = (2f_K + f_\pi)/3$ to quadratic order in the quark masses. The dominant contribution (neglecting corrections $\sim U_\Phi$ and X_Φ^-) is $\tilde{F}_0/f \simeq \varphi_{00}/\overline{\sigma}_0 \simeq 1 - \overline{m}_g^2/\overline{m}_s^2$. We remind the reader that according to our conventions for the definition of Z_Φ, X_Φ^-, etc. the parameter $\tilde{F}_0^2$ corresponds here to a normalization at q_0^2. The value of $\tilde{B}_0$ is related to the proportionality constant[5] a_q in (1.2). It sets the scale for the average current quark mass to lowest order. An estimate of the current quark masses and $\tilde{B}_0$ in the linear meson model can be found in [10].

Finally, it is instructive to relate the fields of the nonlinear sigma model to the linear representations contained in Φ:

$$\Phi = \overline{\sigma}_0 + \frac{1}{\sqrt{2}}\left(i\phi_p + \frac{i}{\sqrt{3}}\chi_p + \phi_s + \frac{1}{\sqrt{3}}\chi_s\right) \,. \tag{3.15}$$

[5] Note that only the combination a_qM_q is independent of the renormalization scale used for the definition of the current quark masses

To lowest order one may use for $\langle \Phi \rangle = \varphi_{00}$ the relations

$$\begin{aligned}
\chi_p &= -\frac{i}{\sqrt{6}}\varphi_{00}\,\mathrm{Tr}\left(U - U^\dagger\right)\\
\Phi_p &= -\frac{i}{\sqrt{2}}\varphi_{00}\left[U - U^\dagger - \frac{1}{3}\,\mathrm{Tr}\left(U - U^\dagger\right)\right]\\
\chi_s &= \frac{1}{\sqrt{6}}\varphi_{00}\left[\mathrm{Tr}\left(U + U^\dagger\right) - 6\right]\\
\Phi_s &= \frac{1}{\sqrt{2}}\varphi_{00}\left[U + U^\dagger - \frac{1}{3}\,\mathrm{Tr}\left(U + U^\dagger\right)\right]
\end{aligned} \tag{3.16}$$

which implies for the kinetic terms

$$\begin{aligned}
\mathrm{Tr}\,\partial^\mu\Phi_p\partial_\mu\Phi_p &= \frac{1}{2}\varphi_{00}^2\Big\{2\,\mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U - \mathrm{Tr}\,\partial^\mu U\partial_\mu U\\
&\quad - \mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U^\dagger + \frac{1}{3}\left[\mathrm{Tr}(\partial_\mu U - \partial_\mu U^\dagger)\right]^2\Big\}\\
\partial^\mu\chi_p\partial_\mu\chi_p &= -\frac{1}{6}\varphi_{00}^2\left[\mathrm{Tr}(\partial_\mu U - \partial_\mu U^\dagger)\right]^2\\
\mathrm{Tr}\,\partial^\mu\Phi_s\partial_\mu\Phi_s &= \frac{1}{2}\varphi_{00}^2\Big\{2\,\mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U + \mathrm{Tr}\,\partial^\mu U\partial_\mu U\\
&\quad + \mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U^\dagger - \frac{1}{3}\left[\mathrm{Tr}(\partial_\mu U + \partial_\mu U^\dagger)\right]^2\Big\}\\
\partial^\mu\chi_s\partial_\mu\chi_s &= \frac{1}{6}\varphi_{00}^2\left(\mathrm{Tr}(\partial_\mu U + \partial_\mu U^\dagger)\right)^2 .
\end{aligned} \tag{3.17}$$

On the other hand, expanding U for small Π_z and ϑ yields

$$\begin{aligned}
\varphi_{00}\tilde{U} &= \varphi_{00} + \frac{i}{2}Z_m^{-\frac{1}{2}}\lambda_z\Pi_z - \frac{1}{8}Z_m^{-1}\frac{(\lambda_z\Pi_z)^2}{\varphi_{00}} + \ldots\\
\varphi_{00}U &= \varphi_{00}\tilde{U} - \frac{i}{3}\varphi_{00}\vartheta - \frac{1}{18}\varphi_{00}\vartheta^2\\
&\quad + \frac{1}{6}Z_m^{-\frac{1}{2}}\varphi_{00}\vartheta\lambda_z\Pi_z + \ldots .
\end{aligned} \tag{3.18}$$

To linear order in Π and ϑ one can identify Π with $m = (2Z_m)^{1/2}\phi_p$

$$\Pi = \lambda_z\Pi_z = (2Z_m)^{\frac{1}{2}}\Phi_p = \lambda_z m_z \tag{3.19}$$

and ϑ is proportional to $p = Z_p^{1/2}\chi_p$

$$\vartheta = -\frac{3}{\sqrt{6}}\frac{\chi_p}{\varphi_{00}} = -\frac{3}{\sqrt{6}}Z_p^{-\frac{1}{2}}\frac{p}{\varphi_{00}} . \tag{3.20}$$

4 Next to leading order quark mass expansion

Going beyond the lowest order in chiral perturbation theory requires a solution for $S[U]$ in the presence of "sources" involving quark masses, non–vanishing ϑ and derivatives of U. We will construct an iterative solution $S[U]$ treating the sources as small corrections. Inserting this solution into the action will produce a systematic expansion of the effective action of the nonlinear sigma model. The lowest order of this expansion is $S = \varphi_{00}$ and has been discussed in the previous section. Next one expands the general solution for $U \neq 1$, namely, $S = \varphi_{00} + S_1 + \ldots \equiv \varphi_{00} + H + T + \ldots$, with $\mathrm{Tr}\,H = 0$ and $T = (\mathrm{Tr}\,S_1)/3$, while keeping only terms linear in S_1 and the sources. To this order one needs the Lagrangian for H and T in the approximation

$$\begin{aligned}
\mathcal{L}^{(H,T)} &= \frac{\overline{M}_H^2}{2}\,\mathrm{Tr}\,H^2 - \mathrm{Tr}\,A^{(H)}[U]H\\
&\quad + \frac{\overline{M}_T^2}{2}T^2 - A^{(T)}[U]T
\end{aligned} \tag{4.1}$$

The mass terms $\overline{M}_H^2$ and $\overline{M}_T^2$ obtain from the second derivatives of V at $S = \varphi_{00}$ (to leading order we may set $\varphi_{00} = \overline{\sigma}_0$ here):

$$\overline{M}_H^2 = 2\overline{m}_g^2 + 2\overline{\nu}\,\overline{\sigma}_0 + 6\overline{\lambda}_2\overline{\sigma}_0^2 \tag{4.2}$$
$$\overline{M}_T^2 = 6\overline{m}_g^2 - 3\overline{\nu}\,\overline{\sigma}_0 + 36\overline{\lambda}_1\overline{\sigma}_0^2 + 36\overline{\beta}_1\overline{\sigma}_0^3 + 36\overline{\beta}_4\overline{\sigma}_0^4 = 6\overline{m}_s^2 .$$

The source terms have three different contributions. The first arises from the quark mass term

$$\begin{aligned}
A_\jmath^{(H)} &= \frac{1}{2}\left(U\jmath^\dagger + \jmath U^\dagger\right) - \frac{1}{6}\,\mathrm{Tr}\left(U\jmath^\dagger + \jmath U^\dagger\right)\\
A_\jmath^{(T)} &= \frac{1}{2}\,\mathrm{Tr}\left(U\jmath^\dagger + \jmath U^\dagger\right) .
\end{aligned} \tag{4.3}$$

The second contribution obtains from linearizing the kinetic term in $H + T$

$$\begin{aligned}
A_{\mathrm{kin}}^{(H)} &= -2\varphi_{00}\left(Z_\Phi + 2X_\Phi^-\varphi_{00}^2\right)\\
&\times\left(\partial_\mu U\partial^\mu U^\dagger - \frac{1}{3}\,\mathrm{Tr}\,\partial_\mu U\partial^\mu U^\dagger\right)\\
&-2iU_\Phi\varphi_{00}^2\left(\partial_\mu\left[\sin\vartheta U\partial^\mu U^\dagger\right] - \frac{i}{3}\partial_\mu\left[\sin\vartheta\partial^\mu\vartheta\right]\right)\\
A_{\mathrm{kin}}^{(T)} &= -\varphi_{00}\left\{2\left(Z_\Phi + 2X_\Phi^-\varphi_{00}^2\right) + 3U_\Phi\varphi_{00}\cos\vartheta\right\}\\
&\times\mathrm{Tr}\,\partial_\mu U\partial^\mu U^\dagger\\
&-\varphi_{00}^2\left\{12\varphi_{00}^3\left(V_\Phi\cos^2\vartheta + \tilde{V}_\Phi\sin^2\vartheta\right) + U_\Phi\cos\vartheta\right\}\\
&\times\partial_\mu\vartheta\partial^\mu\vartheta\\
&-4\varphi_{00}^2\left\{3\varphi_{00}^3\sin\vartheta\cos\vartheta\left(V_\Phi - \tilde{V}_\Phi\right) + U_\Phi\sin\vartheta\right\}\\
&\times\partial^\mu\partial_\mu\vartheta .
\end{aligned} \tag{4.4}$$

Finally, the last term results from the ϑ–dependence of V induced by the invariant ξ (with φ_{00} replaced by $\overline{\sigma}_0$ here)

$$\begin{aligned}
A_\vartheta^{(H)} &= 0\\
A_\vartheta^{(T)} &= -3\overline{\nu}\,\overline{\sigma}_0^2\left(1 - \cos\vartheta\right) .
\end{aligned} \tag{4.5}$$

We point out that we have neglected in (4.1) kinetic terms for H and T as induced by $\mathcal{L}_{\mathrm{kin}}$. This is consistent with our approximation since an expansion of the propagator

$(\overline{M}^2 + Zq^2)^{-1}$ in powers of the momentum squared q^2 produces higher derivatives.

The solution

$$\begin{aligned} H &= \overline{M}_H^{-2} A^{(H)}[U] \\ T &= \overline{M}_T^{-2} A^{(T)}[U] \end{aligned} \tag{4.6}$$

has now to be reinserted into the effective action. This yields the next to leading order corrections to the effective Lagrangian of the nonlinear sigma model

$$\begin{aligned} \mathcal{L}^{(1)}_{\chi PT} = &-\frac{1}{2\overline{M}_H^2} \operatorname{Tr}\left(A_\jmath^{(H)} + A_{\text{kin}}^{(H)} + A_\vartheta^{(H)}\right)^2 \\ &-\frac{1}{2\overline{M}_T^2}\left(A_\jmath^{(T)} + A_{\text{kin}}^{(T)} + A_\vartheta^{(T)}\right)^2 . \end{aligned} \tag{4.7}$$

We will write the terms which do not involve ϑ explicitly (i.e. beyond the implicit ϑ–dependence of U, $U^\dagger$) in a notation analogous to that of [2] (for Euclidean space–time)

$$\begin{aligned} \mathcal{L}^{(1)}_{\chi\text{PT}} = &-\hat{L}_1 \left[\operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2 \\ &- \hat{L}_2 \operatorname{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \operatorname{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right) \\ &- \hat{L}_3 \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)^2 \\ &+ 2B_0 \hat{L}_4 \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) \operatorname{Tr} M_q \left(U^\dagger + U\right) \\ &+ 2B_0 \hat{L}_5 \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger \left[M_q U^\dagger + U M_q\right]\right) \\ &- 4B_0^2 \hat{L}_6 \left[\operatorname{Tr} M_q \left(U^\dagger + U\right)\right]^2 \\ &- 4B_0^2 \hat{L}_7 \left[\operatorname{Tr} M_q \left(U^\dagger - U\right)\right]^2 \\ &- 4B_0^2 \hat{L}_8 \operatorname{Tr}\left(M_q U^\dagger M_q U^\dagger + M_q U M_q U\right) . \end{aligned} \tag{4.8}$$

The coefficients $\hat{L}_i$ can be read off from (4.7) by setting $\vartheta = 0$. The next to leading order quark mass corrections in the potential for U arise from the terms $\sim A_\jmath^2$

$$\begin{aligned} \mathcal{L}^{(1)}_m = &-\frac{1}{8\overline{M}_H^2}\left(\operatorname{Tr} \jmath^\dagger U \jmath^\dagger U + \operatorname{Tr} U^\dagger \jmath U^\dagger \jmath\right) \\ &-\frac{1}{8}\left(\frac{1}{\overline{M}_T^2} - \frac{1}{3\overline{M}_H^2}\right)\left[\operatorname{Tr}\left(\jmath^\dagger U + U^\dagger \jmath\right)\right]^2 . \end{aligned} \tag{4.9}$$

Comparing with (4.8) we infer the constants

$$\begin{aligned} \hat{L}_6 &= \frac{1}{2}\left(Z_\Phi + U_\Phi \varphi_{00} + X_\Phi^- \varphi_{00}^2\right)^2 \left(\frac{\varphi_{00}^2}{\overline{M}_T^2} - \frac{\varphi_{00}^2}{3\overline{M}_H^2}\right) \\ \hat{L}_7 &= 0 \\ \hat{L}_8 &= \frac{1}{2}\left(Z_\Phi + U_\Phi \varphi_{00} + X_\Phi^- \varphi_{00}^2\right)^2 \frac{\varphi_{00}^2}{\overline{M}_H^2} . \end{aligned} \tag{4.10}$$

Quark mass corrections to the kinetic terms are induced by $A_\jmath A_{\text{kin}}$ and read

$$\begin{aligned} \mathcal{L}^{(1)}_{\text{kin,m}} = &\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right) \frac{\varphi_{00}}{\overline{M}_H^2} \\ &\times \operatorname{Tr}\left[\left(U\jmath^\dagger + \jmath U^\dagger\right) \partial_\mu U \partial^\mu U^\dagger\right] \\ &+ \left[\left(Z_\Phi + \frac{3}{2} U_\Phi \varphi_{00} + 2X_\Phi^- \varphi_{00}^2\right) \frac{\varphi_{00}}{\overline{M}_T^2} \right. \\ &\left. - \frac{1}{3}\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right) \frac{\varphi_{00}}{\overline{M}_H^2}\right] \\ &\times \operatorname{Tr}\left(U\jmath^\dagger + \jmath U^\dagger\right) \operatorname{Tr} \partial_\mu U \partial^\mu U^\dagger . \end{aligned} \tag{4.11}$$

From there we find the coefficients

$$\begin{aligned} \hat{L}_4 = &\, 2\left(Z_\Phi + U_\Phi \varphi_{00} + X_\Phi^- \varphi_{00}^2\right) \\ &\times \left[\left(Z_\Phi + \frac{3}{2} U_\Phi \varphi_{00} + 2X_\Phi^- \varphi_{00}^2\right) \frac{\varphi_{00}^2}{\overline{M}_T^2} \right. \\ &\left. - \frac{1}{3}\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right) \frac{\varphi_{00}^2}{\overline{M}_H^2}\right] \end{aligned} \tag{4.12}$$

$$\hat{L}_5 = 2\left(Z_\Phi + U_\Phi \varphi_{00} + X_\Phi^- \varphi_{00}^2\right)\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right) \frac{\varphi_{00}^2}{\overline{M}_H^2} .$$

Finally, we turn to the terms involving four derivatives of U. In the linear sigma model they obtain also contributions from invariants involving four derivatives of Φ. For instance, a term of the form

$$\mathcal{L} = \overline{H}_\Phi \operatorname{Tr}\left(\partial^\mu \Phi^\dagger \partial_\mu \Phi\right)^2 \tag{4.13}$$

yields

$$\mathcal{L}^{(0)}_{\text{kin},4} = \overline{H}_\Phi \varphi_{00}^4 \operatorname{Tr}\left(\partial^\mu U^\dagger \partial_\mu U\right)^2 . \tag{4.14}$$

Since we are considering a linear meson model without explicit vector and axial–vector fields it is plausible that the dominant part of $\overline{H}_\Phi$ and also $\hat{L}_3$ (as well as $\hat{L}_1$ and $\hat{L}_2$) is generated by the exchange of vector mesons [12]. Additional contributions are produced by scalar exchange $\sim A_{\text{kin}}^2$:

$$\begin{aligned} \mathcal{L}^{(1)}_{\text{kin},4} = &-2\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right)^2 \frac{\varphi_{00}^2}{\overline{M}_H^2} \\ &\times \left\{\operatorname{Tr}\left(\partial^\mu U \partial_\mu U^\dagger\right)^2 - \frac{1}{3}\left[\operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2\right\} \\ &-2\left(Z_\Phi + \frac{3}{2} U_\Phi \varphi_{00} + 2X_\Phi^- \varphi_{00}^2\right)^2 \\ &\times \frac{\varphi_{00}^2}{\overline{M}_T^2}\left[\operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2 . \end{aligned} \tag{4.15}$$

leading to $\hat{L}_i$ are

$$\begin{aligned} \hat{L}_1^{(H,T)} = &\, 2\left(Z_\Phi + \frac{3}{2} U_\Phi \varphi_{00} + 2X_\Phi^- \varphi_{00}^2\right)^2 \frac{\varphi_{00}^2}{\overline{M}_T^2} \\ &- \frac{2}{3}\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right)^2 \frac{\varphi_{00}^2}{\overline{M}_H^2} \\ \hat{L}_3^{(H)} = &\, 2\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right)^2 \frac{\varphi_{00}^2}{\overline{M}_H^2} . \end{aligned} \tag{4.16}$$

Finally, one may also extract the interactions between the pseudoscalar Goldstone bosons and the pseudoscalar singlet ϑ to linear order in ϑ. Taking the singlet–octet mixing into account they provide the vertices which are needed for a computation of the decay widths for $\eta' \to \eta\pi\pi$ and $\eta' \to 3\pi_0$. Beyond those terms arising from the implicit ϑ–dependence of $\mathcal{L}^{(0)}_{\chi PT} + \mathcal{L}^{(1)}_{\chi PT}$, (3.9), (4.8), there are also explicit ϑ–dependencies appearing in (4.7). They are given to linear order in ϑ by

$$\begin{aligned}\mathcal{L}^{(1)}_{\chi PT,\vartheta} = -iU_\Phi \frac{\varphi_{00}^2}{\overline{M}_H^2}\vartheta \Big\{ &\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger \left(U\jmath^\dagger - \jmath U^\dagger\right)\right] \\ &- 4\varphi_{00}\left(Z_\Phi + 2X_\Phi^- \varphi_{00}^2\right) \\ &\times \mathrm{Tr}\left[\partial_\mu\left(\partial_\nu U \partial^\nu U^\dagger\right) U \partial^\mu U^\dagger\right]\Big\} \,. \end{aligned} \tag{4.17}$$

5 Integrating out the η' meson

If one is only interested in the effective interactions of the pseudoscalar octet mesons one may integrate out also the field ϑ which corresponds dominantly to the η' particle. The procedure parallels the one of the preceding section. The mass term $\overline{M}^2_\vartheta$ is given by (3.7), and the linear terms involving $\tilde{U}$, i.e. $-A^{(\vartheta)}[\tilde{U}]\vartheta$ give rise to effective contributions to the low energy constants for $\tilde{U}$

$$\Delta\mathcal{L}^{(1,\vartheta)}_{\chi PT} = -\frac{1}{2\overline{M}^2_\vartheta}\left(A^{(\vartheta)}_\jmath + A^{(\vartheta)}_{\mathrm{kin}}\right)^2 \,. \tag{5.1}$$

For ϑ integrated out and $U \equiv \tilde{U}$ we denote the constants appearing in the analog of (4.8) by $\tilde{L}_i$. The difference between the $\tilde{L}_i$ and $\hat{L}_i$ can be computed from (5.1) with

$$\begin{aligned} A^{(\vartheta)}_\jmath &= \frac{i}{6}\varphi_{00}\,\mathrm{Tr}\left(\tilde{U}^\dagger\jmath - \jmath^\dagger\tilde{U}\right) \\ A^{(\vartheta)}_{\mathrm{kin}} &= 0 \,. \end{aligned} \tag{5.2}$$

Here we have omitted terms resulting from $\mathcal{L}^{(1)}_{\chi PT} + \mathcal{L}^{(1)}_{\chi PT,\vartheta}$ which do not contribute to the $\tilde{L}_i$ since they are of higher order in the quark mass expansion. One therefore finds

$$\Delta\mathcal{L}^{(1,\vartheta)}_{\chi PT} = \frac{\varphi_{00}^2}{72\overline{M}^2_\vartheta}\left[\mathrm{Tr}\left(\tilde{U}^\dagger\jmath - \jmath^\dagger\tilde{U}\right)\right]^2 \,. \tag{5.3}$$

The dominant contribution of this term is an effective additional mass term for the η meson which is due to η–η' mixing. Equation (5.3) yields

$$\tilde{L}_7 = -\frac{1}{18}\left(Z_\Phi + U_\Phi\varphi_{00} + X_\Phi^-\varphi_{00}^2\right)^2 \frac{\varphi_{00}^4}{\overline{M}^2_\vartheta} \,. \tag{5.4}$$

For the remaining $\tilde{L}_i$ there is no contribution from pseudoscalar singlet exchange such that $\tilde{L}_i = \hat{L}_i$ for $i \neq 7$.

6 Estimates of the low energy constants

Having established in the preceding sections the formal correspondence between the parameters of the linear meson model and the effective couplings $\tilde{L}_i$ of the nonlinear sigma model we may estimate the latter using the phenomenological information about the linear meson model as input. The experimental input for the linear meson model considered in [5] concerns only several (pseudo–)scalar meson masses and decay constants as well as the η–η' mixing angle. A comparison with the values of the low energy constants L_i of chiral perturbation theory (see for instance [14]) therefore implicitly relates different experimental observations. We remind the reader at this point that our values for the low energy constants are effective values which result in chiral perturbation theory after computing the loop corrections from pseudoscalar fluctuations. We also note that some of these constants involve as an important parameter the mass of the scalar singlet (the σ particle) which was not determined by the analysis of [5].

We will start by expressing the $\tilde{L}_i$ determined in the preceding sections in terms of the parameters introduced in [5]. We can use here the approximations $\varphi_{00} = \overline{\sigma}_0 \equiv Z_m^{-1/2}\sigma_0$, $\nu = Z_m^{-3/2}\overline{\nu}$ which yields

$$\begin{aligned} \overline{M}^2_H &= 2Z_h m_h^2 \\ \overline{M}^2_T &= 6Z_s m_s^2 \\ \overline{M}^2_\vartheta &= \frac{2}{3}\frac{Z_p}{Z_m}\sigma_0^2 m_p^{(0)2} = \nu\sigma_0^3 \end{aligned}$$

$$\begin{aligned} &Z_\Phi + U_\Phi\varphi_{00} + X_\Phi^-\varphi_{00}^2 = Z_m \\ &Z_\Phi + 2X_\Phi^-\varphi_{00}^2 = Z_m\left\{1 + \left(\frac{Z_h}{Z_m}\right)^{\frac{1}{2}}\omega_m\sigma_0\right\} \equiv Z_m\beta \\ &Z_\Phi + \frac{3}{2}U_\Phi\varphi_{00} + 2X_\Phi^-\varphi_{00}^2 = \\ &\quad Z_m\left\{1 + \left(\frac{Z_h}{Z_m}\right)^{\frac{1}{2}}\omega_m\sigma_0 + \frac{1}{2}\left(1 - \frac{Z_p}{Z_m}\right)\right\} \equiv Z_m\alpha \,. \end{aligned} \tag{6.1}$$

Here m_s denotes the mass of the sigma resonance and m_h is the average mass of the scalar octet with Z_s and Z_h denoting the corresponding wave function renormalization constants, respectively. The parameters α and β depend on the quantity ω_m [5] which plays an important role in the η–η' mixing and can be determined phenomenologically from the decays $\eta \to 2\gamma$ and $\eta' \to 2\gamma$. We obtain

$$\begin{aligned} \tilde{L}_1^{(H,T)} &= \frac{1}{3}\left(\alpha^2\frac{Z_m}{Z_s}\frac{\sigma_0^2}{m_s^2} - \beta^2\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2}\right) \\ \tilde{L}_2^{(H,T)} &= 0 \\ \tilde{L}_3^{(H)} &= \beta^2\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2} \end{aligned}$$

$$\begin{aligned}
\tilde{L}_4 &= \frac{1}{3}\left(\alpha\frac{Z_m}{Z_s}\frac{\sigma_0^2}{m_s^2} - \beta\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2}\right) \\
&\equiv \frac{1}{3}\alpha\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2}\left(\frac{\alpha-\beta}{\alpha}+\gamma\right) \qquad (6.2) \\
\tilde{L}_5 &= \beta\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2} \\
\tilde{L}_6 &= \frac{1}{12}\left(\frac{Z_m}{Z_s}\frac{\sigma_0^2}{m_s^2} - \frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2}\right) \equiv \frac{1}{12}\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2}\gamma \\
\tilde{L}_7 &= -\frac{1}{12}\frac{Z_m}{Z_p}\frac{\sigma_0^2}{m_p^{(0)2}} = -\frac{1}{18}\frac{\sigma_0}{\nu} \\
\tilde{L}_8 &= \frac{1}{4}\frac{Z_m}{Z_h}\frac{\sigma_0^2}{m_h^2}\ . \qquad (6.3)
\end{aligned}$$

From [5] we infer the values

$$\begin{aligned}
\alpha &= 0.55 \pm 0.04 \\
\beta &= 0.53 \pm 0.02 \\
\sigma_0 &= (53.9 \pm 0.2)\,\mathrm{MeV} \\
\nu &= (6410 \pm 420)\ \mathrm{MeV} \\
\left(\frac{Z_h}{Z_m}\right)^{1/2} m_h &= (825 \pm 10)\,\mathrm{MeV}\ . \qquad (6.4)
\end{aligned}$$

The mass m_s of the σ meson as well as the quotient Z_s/Z_m remained undetermined in [5]. We have parameterized this uncertainty in (6.2) with the scalar singlet–octet splitting

$$\gamma \equiv \frac{Z_h}{Z_s}\frac{m_h^2}{m_s^2} - 1 \qquad (6.5)$$

such that presumably $|\gamma| < 1$. (In fact, to leading order in the large–N_c expansion one has exactly $\gamma = 0$.) This leads to the results in Table 6 where we have omitted our numerical results for $\tilde{L}_1^{(H,T)}$, $\tilde{L}_2^{(H,T)}$ and $\tilde{L}_3^{(H)}$ since they are incomplete without the contributions from higher derivative terms in the linear meson model. For comparison the table also gives recent phenomenological estimates [14] of chiral perturbation theory. The $\tilde{L}_i$ are here compared with the couplings $L_i(\mu)$ of the nonlinear sigma model normalized at scales $\mu = 400\,\mathrm{MeV}$ (line (b)) and $\mu = 600\,\mathrm{MeV}$ (line (c)). The overall agreement of the linear meson model estimates with these values is striking. In particular, we may use the low energy constants L_5 and L_8 for an estimate of β in chiral perturbation theory:

$$\beta_{\chi PT} = \frac{1}{4}\frac{L_5}{L_8} = 0.5 \pm 0.3\ . \qquad (6.6)$$

This agrees well with the value given in (6.4) which was extracted in [5] essentially from the ratio of decay rates $\eta \to 2\gamma$ vs. $\eta' \to 2\gamma$. We emphasize that the difference of α and β from one is entirely due to the non–minimal kinetic terms in the linear meson model. A linear model that only includes renormalizable interactions would be in very poor agreement with phenomenology, as already noted in [15]. The difference between our values for $\tilde{L}_5$ and $\tilde{L}_8$ and the central ones for $L_5(\mu)$ and $L_8(\mu)$ in line (b) of Table 6, respectively, extracted from [14] could easily be absorbed by a somewhat higher choice of the renormalization scale μ of chiral perturbation theory as demonstrated in line (c) of Table 6. On the other hand, relatively large higher order quark mass effects were observed in the computation of $f_K - f_\pi$ in the linear meson model [5] and these effects are not included in chiral perturbation theory.

It should be noted that the errors of the $\tilde{L}_i$ given in line (a) of Table 6 only reflect uncertainties in the determination of the parameters (6.4) within the linear meson model. They do not represent a systematic error analysis but rather follow from the scattering of values quoted for a range of realistic assumptions in the tables of [5]. We believe, nevertheless, that these uncertainties are not too far from realistic errors once the information on higher quark mass corrections and propagator effects also contained in the linear meson model is properly included. In fact, the replacement $\varphi_{00} \to \overline{\sigma}_0$ in our computation of the $\tilde{L}_i$ already includes resummed higher order contributions from the singlet mass term $(m_u + m_d + m_s)/3$.

Our result for the scalar exchange contribution $\tilde{L}_3^{(H)} = (1.29 \pm 0.14) \cdot 10^{-3}$ has opposite sign and is much smaller than the value $L_3 = -(4.4 \pm 2.5) \cdot 10^{-3}$ given in [14]. We take this as a strong indication that kinetic terms with four derivatives in the linear meson model are dominant as suggested by vector meson exchange dominance [12]. This is further substantiated by the fact that there is no contribution to $\tilde{L}_2$ due to scalar meson or η' exchange in contrast to the non–vanishing value $L_2 = (1.7 \pm 0.7) \cdot 10^{-3}$ obtained from D–wave $\pi\pi$ scattering [14]. Nevertheless, we wish to note that our results (6.2) for $\tilde{L}_1^{(H,T)}$, $\tilde{L}_2^{(H,T)}$ and $\tilde{L}_3^{(H)}$, i.e. for the scalar meson exchange contributions to L_1, L_2 and L_3, respectively, agree qualitatively with those of [12].

7 Conclusions

Within the linear meson model we have estimated the effective couplings $\tilde{L}_4$–$\tilde{L}_8$ of the non–linear sigma model for the light pseudoscalar mesons. They correspond to effective vertices related to interactions appearing in next to leading order chiral perturbation theory. Whereas the low energy couplings L_4–L_8 of chiral perturbation theory are the renormalization scale (and scheme) dependent couplings of a perturbative expansion, the effective couplings $\tilde{L}_i$ already include all quantum fluctuations and correspond to $1PI$ vertices. They are independent of the renormalization scale μ but involve typical momenta q of the fields appearing in the n–point functions. We evaluate the effective kinetic terms at the "pole" for an average pseudoscalar octet mass, i.e., $q_0^2 = -(2M_{K^\pm}^2 + M_{\pi^\pm}^2)/3$. A comparison with the low energy constants $L_i(\mu)$ for $\mu^2 = -q_0^2$ shows good agreement within errors (see Table 6). The central values almost coincide for $\mu \simeq 600\,\mathrm{MeV}$. A computation of the $1PI$ vertices $\tilde{L}_i$ within chiral perturbation theory would be very valuable for a more precise connection between the linear meson model and chiral perturbation theory. In view of the comparatively small uncertain-

Table 1. This table shows our estimates of some of the $\tilde{L}_i$ in line (a) in comparison with the phenomenological results for $L_i(\mu)$ taken at normalization scales $\mu = 400\,\mathrm{MeV}$ (b) and $\mu = 600\,\mathrm{MeV}$ (c)

	$\tilde{L}_4 \cdot 10^3$	$\tilde{L}_5 \cdot 10^3$	$\tilde{L}_6 \cdot 10^3$	$\tilde{L}_7 \cdot 10^3$	$\tilde{L}_8 \cdot 10^3$
(a)	$(0.78 \pm 0.06)(\gamma + 0.04)$	2.26 ± 0.10	$(0.36 \pm 0.01)\gamma$	-0.47 ± 0.03	1.07 ± 0.03
(b)	0.2 ± 0.5	3.0 ± 0.5	0.1 ± 0.3	-0.4 ± 0.15	1.3 ± 0.3
(c)	-0.1 ± 0.5	2.0 ± 0.5	-0.1 ± 0.3	-0.4 ± 0.15	1.1 ± 0.3

ties for the $\tilde{L}_i$ this may lead to a more precise determination of some of the L_i.

In our version of the linear meson model the explicit chiral symmetry breaking due to current quark masses appears only in form of a linear source term. In this approximation the effective couplings $\tilde{L}_4$–$\tilde{L}_8$ are entirely determined by the exchange of scalar 0^{++} octet and singlet states. The hypothesis that these couplings are dominated by scalar exchange has been found earlier [12] to be in agreement with observation. There large–N_c results were employed to estimate L_4, L_6 and L_7 whereas L_5 and L_8 were determined from phenomenological input. We provide here a quantitative framework for a computation of these couplings based on phenomenological estimates of the couplings in the linear meson model. At the present stage the couplings $\tilde{L}_4$ and $\tilde{L}_6$ still depend on the unknown scalar octet–singlet splitting γ, (6.5). It would be very interesting to determine this parameter from phenomenological considerations within the linear meson model. Furthermore there is the prospect that many of the parameters of the linear meson model are actually determined by a partial infrared fixed point behavior [7] of the running couplings in the linear quark meson model. This would lead to a theoretical "prediction" for some of the low energy constants $\tilde{L}_i$ independent of phenomenological input. For the time being $\tilde{L}_5$, $\tilde{L}_7$ and $\tilde{L}_8$ are determined effectively in terms of masses and mixing in the η–η' system and the difference $f_K - f_\pi$. The ratio $\tilde{L}_4/\tilde{L}_6 \simeq 2$ follows without further input parameter.

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